Quantitative Language Automata

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TL;DR

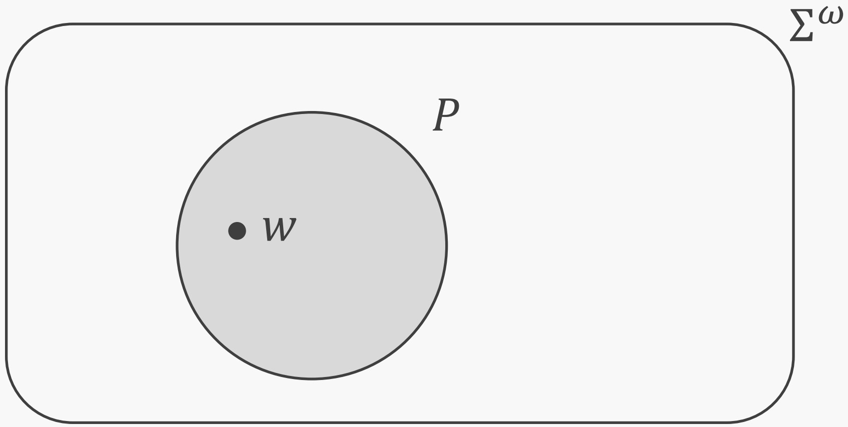
- Quantitative *word* automata (QWAs)
  - $\mathcal{A} : \Sigma^\omega \to \mathbb{D}$
  - $\mathcal{A}(w) = g_{\rho \in Runs(w)} f(\rho)$
- Quantitative *language* automata (QLAs)
  - $\mathbb{A} : 2^{\Sigma^\omega} \to \mathbb{D}$
  - $\mathbb{A}(L) = h_{w \in L}\, g_{\rho \in Runs(w)} f(\rho)$
- QLAs define *quantitative hyperproperties*
  - $\forall/\exists \rightarrow \sup/\inf/\cdots$
- No quantifier alternation
  - QLA problems reduce to QWA problems

Context and Motivation
Boolean Trace Properties

$$P \subseteq \Sigma^\omega$$

For every *execution* of…

- a server, *is it true* that each request is eventually granted?
- a drone, *is it true* that the designated path is never left?
- …



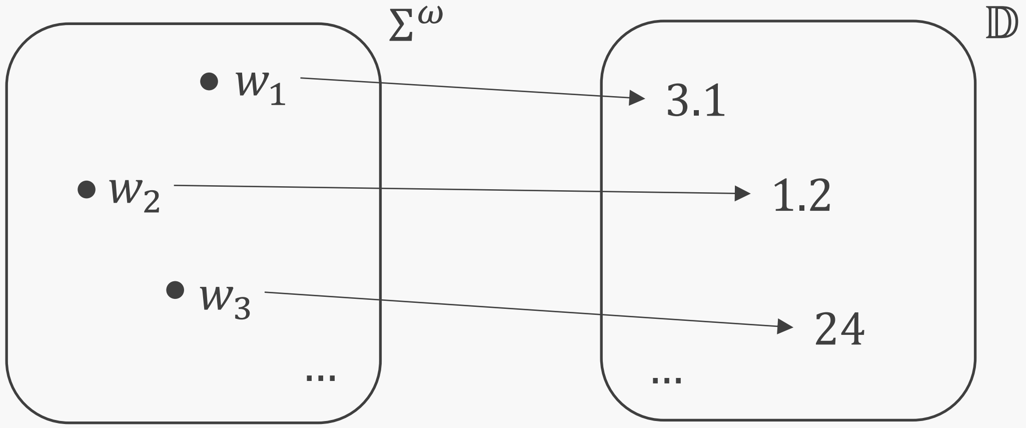
Context and Motivation

Quantitative Trace Properties [CDH08]

$$\Phi : \Sigma^\omega \to \mathbb{D}$$

Given an *execution* of…

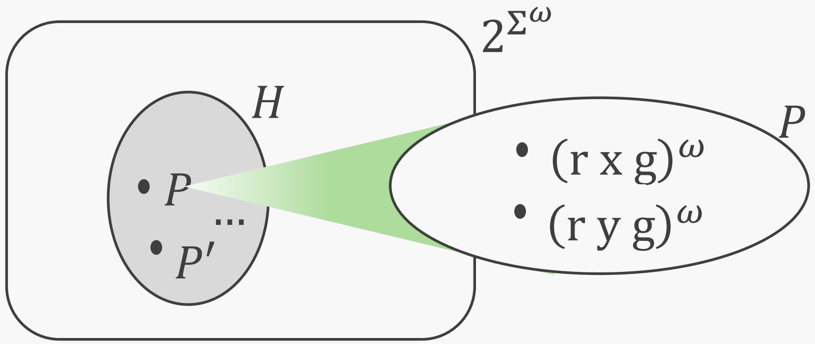
- a server, *what is* its average response time?
- a drone, *what is* the worst-case deviation from its designated path?
- …



Chatterjee, Doyen, and Henzinger, "Quantitative languages", CSL 2008.

Context and Motivation

Boolean Hyperproperties [CS08]



$$H \subseteq 2^{\Sigma^\omega}$$

For every *implementation* of…

- a server, *is it true* that every pair of executions with the same request pattern has the same grant pattern?

$$\forall \pi, \pi' \colon G(\mathrm{r}_\pi \leftrightarrow \mathrm{r}_{\pi'}) \rightarrow G(\mathrm{g}_\pi \leftrightarrow \mathrm{g}_{\pi'})$$

- a drone, *is it true* that for every high-wind execution there is a matching low-wind execution whose paths are ε-close?

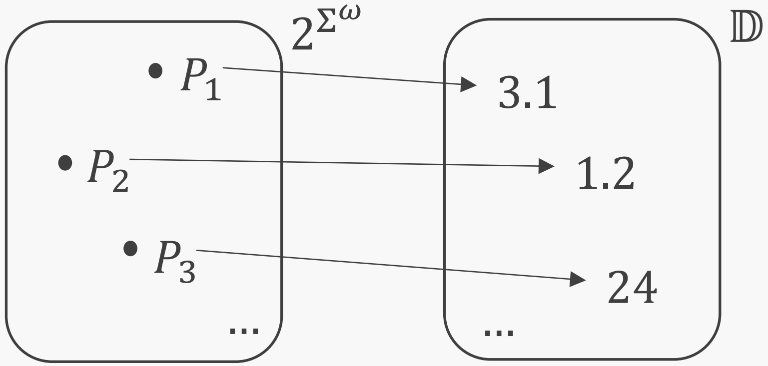
$$\forall \pi \colon G(\mathrm{hiWind}_\pi) \rightarrow \left(\exists \pi' \colon G(\mathrm{loWind}_{\pi'}) \wedge (\mathrm{tgt}_\pi \leftrightarrow \mathrm{tgt}_{\pi'}) \wedge G(\mathrm{close}_\varepsilon(\pi, \pi'))\right)$$

- …

Clarkson and Schneider, "Hyperproperties", CSF 2008.

Context and Motivation

Quantitative Hyperproperties



$$\Theta : 2^{\Sigma^\omega} \to \mathbb{D}$$

Given an *implementation* of…

- a server, *what is* its worst-case average response time?

$$P \mapsto \sup_{w \in P} AvgRespTime(w)$$

- a drone, *what is* the maximal distance between its high-wind runs and their low-wind equivalents?

$$P \mapsto \sup_{w \in P} \inf_{w' \in P} WindDist(w, w')$$

- …

Our Contribution

Quantitative Language Automata (QLAs):
the first specification framework for QHPs of reactive systems

(a)	Evaluation					Nonemptiness		
	$h = g$	$h \neq g$	$h = \mathbb{E}$	$h \in \{\mathsf{Inf}, \mathsf{Sup}\}$	$h = \mathbb{E}$	$h \in \{\mathsf{Inf}, \mathsf{Sup}, \mathbb{E}\}$		
	$\in \{\mathsf{Inf}, \mathsf{Sup}\}$	$\in \{\mathsf{Inf}, \mathsf{Sup}\}$	$g \in \{\mathsf{Inf}, \mathsf{Sup}\}$	$g = \mathbb{E}$	$g = \mathbb{E}$	$g = \mathsf{Sup}$	$g = \mathsf{Inf}$	$g = \mathbb{E}$
$f \in \{\mathsf{Inf}, \mathsf{Sup}, \mathsf{LimInf}, \mathsf{LimSup}\}$	PTIME	PSPACE	EXPTIME	Undecidable	PTIME	PTIME	PSPACE	Undecidable
$f \in \{\mathsf{LimInfAvg}, \mathsf{LimSupAvg}\}$	PTIME	Undecidable	Undecidable	Undecidable	PTIME	PTIME	Undecidable	Undecidable
$f = \mathsf{DSum}$	PTIME	Open-hard	Open-hard	Undecidable	PTIME	PTIME	Open-hard	Undecidable

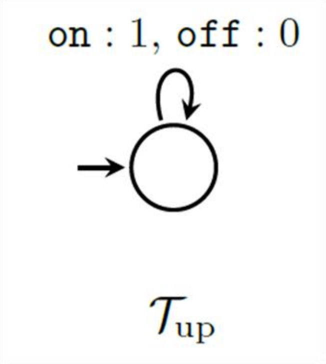
(b)	Evaluation	Nonemptiness	Universality
$h, g, f \in \{\mathsf{Inf}, \mathsf{Sup}, \mathsf{LimInf}, \mathsf{LimSup}\}$ and at least one of h or g is in $\{\mathsf{LimInf}, \mathsf{LimSup}\}$	PSPACE	PSPACE	PSPACE

Quantitative Word Automata (QWAs) [CDH08]

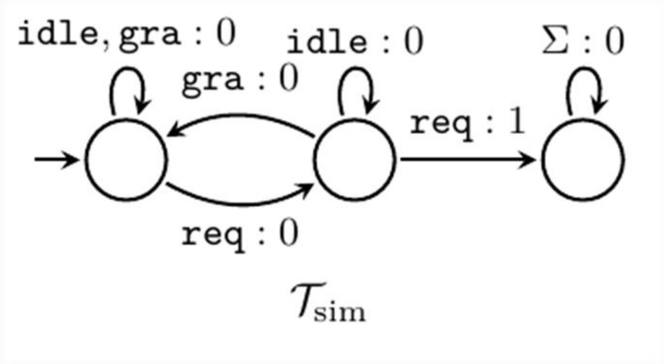
$\mathcal{A} = (g, f\ \mathcal{T})$ $\qquad \mathcal{A}(w) = g_{\rho \in Runs(w,\mathcal{T})} f(\rho)$

- $\mathcal{T}$: weighted labeled finite transition system
- f: run aggregator (a.k.a. value function) in {Inf, Sup, LimInf, LimSup, LimInfAvg, LimSupAvg, DSum}
- g: word aggregator (a.k.a. resolver) in {Inf, Sup, LimInf, LimSup, $\mathbb{E}$}

Example. $\mathcal{A}_1 = (g, \text{LimInfAvg}, \mathcal{T}_{\text{up}})$



Example. $\mathcal{A}_2 = (g, \text{DSum}_2, \mathcal{T}_{\text{sim}})$



Chatterjee, Doyen, and Henzinger, "Quantitative languages", CSL 2008.

Quantitative Language Automata (QLAs)

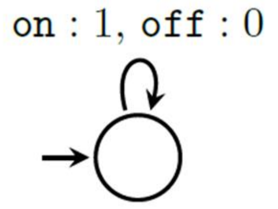
$\mathbb{A} = (h, \mathcal{A})$ $\qquad$ $\mathbb{A}(L) = h_{w \in L} \mathcal{A}(w)$

$\mathcal{A}$ → QWA

h → language aggregator in $\{\mathrm{Inf}, \mathrm{Sup}, \mathrm{LimInf}, \mathrm{LimSup}, \mathbb{E}\}$

Example (worst-case avg. uptime).

$\mathbb{A}_1 = (\mathrm{Inf}, \mathcal{A}_1)$



$\mathcal{A}_1$
(LimInfAvg)

$$\mathbb{A}_1(L) = \inf_{w \in L} \mathcal{A}_1(w)$$

Quantitative Language Automata (QLAs)

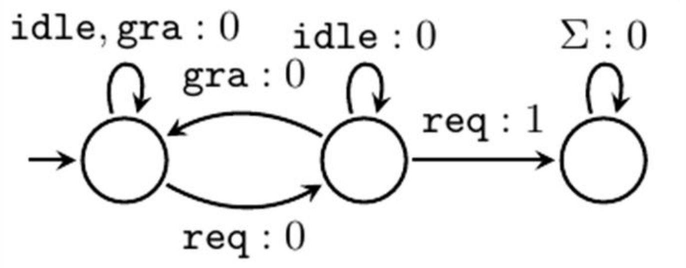
$\mathbb{A} = (h, \mathcal{A})$ $\qquad \mathbb{A}(L) = h_{w \in L} \mathcal{A}(w)$

QWA

language aggregator in $\{\mathrm{Inf}, \mathrm{Sup}, \mathrm{LimInf}, \mathrm{LimSup}, \mathbb{E}\}$

Example (model measuring).

$\mathbb{A}_2 = (\mathrm{Sup}, \mathcal{A}_2)$



$\mathcal{A}_2$
(DSum_2)

$$\mathbb{A}_2(L) = \sup_{w \in L} \mathcal{A}_2(w)$$
$$= \sup_{w \in L} \inf_{w' \in P} d_{Cantor}(w, w')$$

where $P =$ "no two pending req"

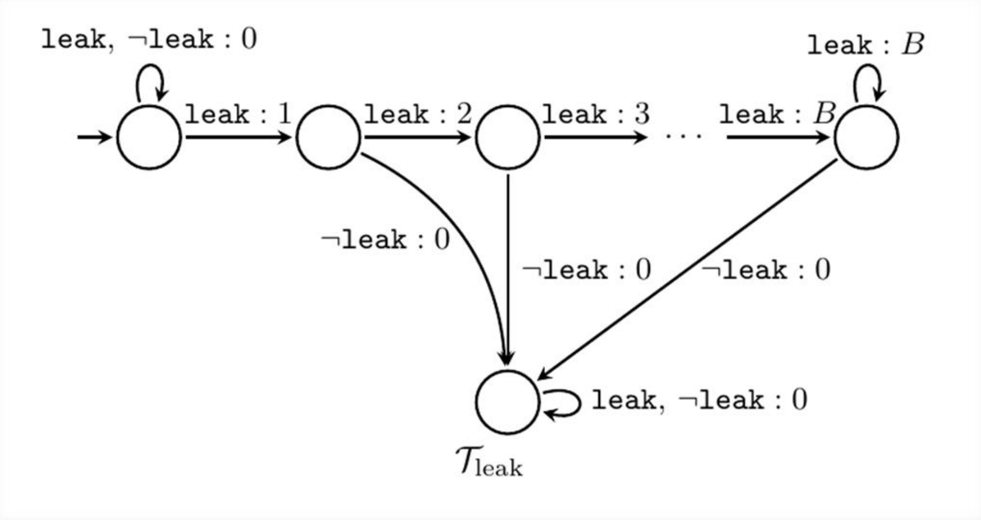
Quantitative Language Automata (QLAs)

Example (longest leaking block).

$$\mathbb{A}_3 = (\mathrm{Sup}, \mathcal{A}_3) \quad \mathcal{A}_3 = (\mathrm{Sup}, \mathrm{Sup}, \mathcal{T}_{\mathrm{leak}})$$

$$\Sigma = 2^{\{lo,hi,out,other\}} \qquad \Sigma' = \Sigma \times \Sigma$$

$$\mathrm{leak}(\sigma, \sigma') \coloneqq (\sigma_{lo} \leftrightarrow \sigma'_{lo}) \land (\sigma_{hi} \not\leftrightarrow \sigma'_{hi}) \land (\sigma_{out} \not\leftrightarrow \sigma'_{out})$$



$\mathcal{A}_3(w)$ = length of the longest leaking block in w (up to B)

$$L \subseteq \Sigma^\omega \to L' \subseteq {\Sigma'}^\omega$$

$$\mathbb{A}_3(L') = \sup_{w \in L'} \mathcal{A}(w) = \sup_{w \in L} \sup_{w' \in L} \mathcal{A}(w, w')$$

Core Problems

Given a QLA $\mathbb{A} = (h, \mathcal{A})$, a threshold $k \in \mathbb{Q}$, and $\sim \in \{>, \geq\}$.

1. **Evaluation**
   Given a nondeterministic Büchi automaton $\mathcal{B}$, compute $\mathbb{A}(L_{\mathcal{B}})$.
2. **Nonemptiness**
   Does $\mathbb{A}(L) \sim k$ hold for some nonempty $L \subseteq \Sigma^\omega$?
3. **Universality**
   Does $\mathbb{A}(L) \sim k$ hold for every nonempty $L \subseteq \Sigma^\omega$?

Evaluation

Given a QLA $\mathbb{A} = (h, \mathcal{A})$ and a nondeterministic Büchi automaton $\mathcal{B}$.

Compute $\mathbb{A}(L_{\mathcal{B}})$.

What happens when the value of an ω-regular language may not be realizable by a lasso word or any word?

Assume $g, h \in \{\mathrm{Inf}, \mathrm{Sup}\}$.

$f \in \cdots$	$\{\mathrm{Inf}, \mathrm{Sup}, \mathrm{LimInf}, \mathrm{LimSup}\}$	$\{\mathrm{LimInfAvg}, \mathrm{LimSupAvg}\}$	$\{\mathrm{DSum}\}$
Evaluation	PTime when $g = h$ PSpace when $g \neq h$	PTime when $g = h$ Incomputable when $g \neq h$	PTime when $g = h$ Open-hard when $g \neq h$

Nonemptiness and Universality

Given a QLA $\mathbb{A} = (h, \mathcal{A})$, a threshold $k \in \mathbb{Q}$, and $\sim \in \{>, \geq\}$.

Does $\mathbb{A}(L) \sim k$ hold for some (resp. every) nonempty $L \subseteq \Sigma^\omega$?

Assume $g = \mathrm{Sup}$ and $h \in \{\mathrm{Inf}, \mathrm{Sup}\}$.

$f \in \cdots$	$\{\mathrm{Inf}, \mathrm{Sup}, \mathrm{LimInf}, \mathrm{LimSup}\}$	$\{\mathrm{LimInfAvg}, \mathrm{LimSupAvg}\}$	$\{\mathrm{DSum}\}$
Nonemptiness	PTime	PTime	PTime
Universality	PSpace	Undecidable	Open-hard

Finite-state Universality

Are extreme values realizable or approximable by lasso words?

$\inf_{w \in \Sigma^\omega} \mathcal{A}(w)$

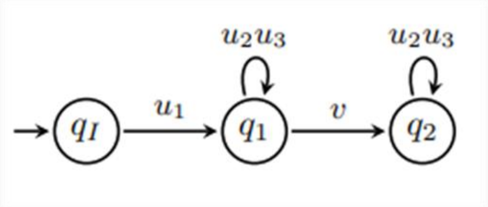
- Consider a QLA $\mathbb{A} = (h, \mathcal{A})$ and a threshold $k \in \mathbb{Q}$.
  If for every $\varepsilon > 0$ there is lasso word w such that $\mathcal{A}(w) < \perp_{\mathcal{A}} + \varepsilon$,
  then $\mathbb{A}(L) \geq k$ for all L iff $\mathbb{A}(L) \geq k$ for all ω-regular L.

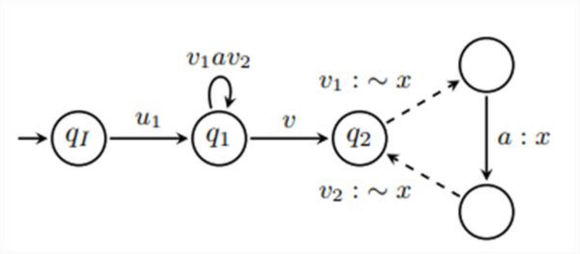
- For every $\varepsilon > 0$ there is lasso word w such that $\mathcal{A}(w) < \perp_{\mathcal{A}} + \varepsilon$.
  - $\mathcal{A} = (\mathrm{Sup}, \mathrm{LimInfAvg}, \mathcal{T})$ False
  - $\mathcal{A} = (\mathrm{Sup}, \mathrm{LimSupAvg}, \mathcal{T})$ True

Limit Aggregators

How to decide if
- a word w has infinitely many runs of value v?
- a language L has infinitely many words of value v?

LimInf and LimSup as word and language aggregators.





Solution: Characterize the patterns in quantitative automata that capture when a value is infinitely realizable (similar to characterizing ambiguity in boolean word automata [WS91, LP18]).

Weber and Seidl, "On the degree of ambiguity of finite automata", TCS, 1991.
Löding and Pirogov, "On finitely ambiguous Büchi automata", DLT 2018.

Conclusion

Quantitative Language Automata (QLAs):
the first specification framework for QHPs of reactive systems

(a)	Evaluation					Nonemptiness		
						$h \in \{\mathsf{Inf}, \mathsf{Sup}, \mathbb{E}\}$		
	$h = g \in \{\mathsf{Inf}, \mathsf{Sup}\}$	$h \neq g \in \{\mathsf{Inf}, \mathsf{Sup}\}$	$h = \mathbb{E}$, $g \in \{\mathsf{Inf}, \mathsf{Sup}\}$	$h \in \{\mathsf{Inf}, \mathsf{Sup}\}$, $g = \mathbb{E}$	$h = \mathbb{E}$, $g = \mathbb{E}$	$g = \mathsf{Sup}$	$g = \mathsf{Inf}$	$g = \mathbb{E}$
$f \in \{\mathsf{Inf}, \mathsf{Sup}, \mathsf{LimInf}, \mathsf{LimSup}\}$	PTime	PSpace	ExpTime	Undecidable	PTime	PTime	PSpace	Undecidable
$f \in \{\mathsf{LimInfAvg}, \mathsf{LimSupAvg}\}$	PTime	Undecidable	Undecidable	Undecidable	PTime	PTime	Undecidable	Undecidable
$f = \mathsf{DSum}$	PTime	Open-hard	Open-hard	Undecidable	PTime	PTime	Open-hard	Undecidable

(b)	Evaluation	Nonemptiness	Universality
$h, g, f \in \{\mathsf{Inf}, \mathsf{Sup}, \mathsf{LimInf}, \mathsf{LimSup}\}$ and at least one of h or g is in $\{\mathsf{LimInf}, \mathsf{LimSup}\}$	PSpace	PSpace	PSpace

Future Work

- Quantifier alternation
- Alternative run aggregators

Thank you!

Related Work

- Robustness semantics of HyperSTL [NKJ+17]
  - $\forall v\colon \exists v'\colon G_{[0,2]}(v > 1) \rightarrow F_{[1,2]}(v' > 2)$
  - Only robustness
- HyperLTL with counting [FHT19]
  - $\forall \pi\colon \#\sigma.\, O\colon G(\pi =_I \sigma) \leq 2^c$
  - Expressively equivalent to HyperLTL
- Quantitative reasoning with predicate transformers [ZZK+24]
  - $\mathrm{whp}\, [C_{qif}](\mathbb{f}_{\ell=x}(h))$
  - Only for finite executions

Nguyen, Kapinski, Jin, Deshmukh, and Johnson, "Hyperproperties of real-valued signals", MEMOCODE 2017.
Finkbeiner, Hahn, and Torfah, "Model checking quantitative hyperproperties", CAV 2019.
Zhang, Zilberstein, Kaminski, and Silva, "Quantitative weakest hyper pre", OOPSLA 2024.

Core Challenges

1. **Evaluation**
   What happens when the value of an ω-regular language may not be realizable by a lasso word or any word?
2. **Finite-state nonemptiness/universality**
   Are extreme values realizable or approximable by lasso word?
3. **Limit word and language aggregators** $\mathrm{g}, \mathrm{h} \in \{\mathrm{LimInf}, \mathrm{LimSup}\}$
   How to decide if a word w has infinitely many runs of value v or a language L has infinitely many words of value v?

Core Problems

▸ **Proposition 4.2.** *Consider a QLA* $\mathbb{A} = (h, \mathcal{A})$ *with* $h \in \{\mathsf{Inf}, \mathsf{Sup}, \mathbb{E}\}$*, a rational* $k \in \mathbb{Q}$*, and* $\triangleright \in \{>, \geq\}$*.*

(a) *Unrestricted variant*

(i) *If* $h \neq \mathsf{Sup}$ *or* $\triangleright = >$*, then* $\mathbb{A}$ *is* $\triangleright$*-nonempty for* k *iff* $\mathcal{A}$ *is* $\triangleright$*-nonempty for* k*.*

(ii) *If* $h = \mathsf{Sup}$*, then* $\mathbb{A}$ *is* $\geq$*-nonempty for* k *iff* $\mathcal{A}$ *is approximate-nonempty for* k*.*

(b) *Finite-state variant*

(i) *If* $h = \mathsf{Sup}$*, then* $\mathbb{A}$ *is finite-state* $\triangleright$*-nonempty for* k *iff* $\mathbb{A}$ *is* $\triangleright$*-nonempty for* k*.*

(ii) *If* $h = \mathsf{Inf}$*, then* $\mathbb{A}$ *is finite-state* $\triangleright$*-nonempty for* k *iff* $\mathcal{A}$ *is lasso-word* $\triangleright$*-nonempty for* k*.*

(iii) *If* $\top_{\mathcal{A}}$ *is achievable by a lasso word, then* $\mathbb{A}$ *is finite-state* $\triangleright$*-nonempty for* k *iff* $\mathbb{A}$ *is* $\triangleright$*-nonempty for* k *iff* $\mathcal{A}$ *is lasso-word* $\triangleright$*-nonempty for* k*.*

For universality, we have the duals where we exchange Inf *with* Sup*,* $>$ *with* $\geq$ *, nonempty with universal, approximate-nonempty with approximate-universal, and* $\top_{\mathcal{A}}$ *with* $\bot_{\mathcal{A}}$*.*

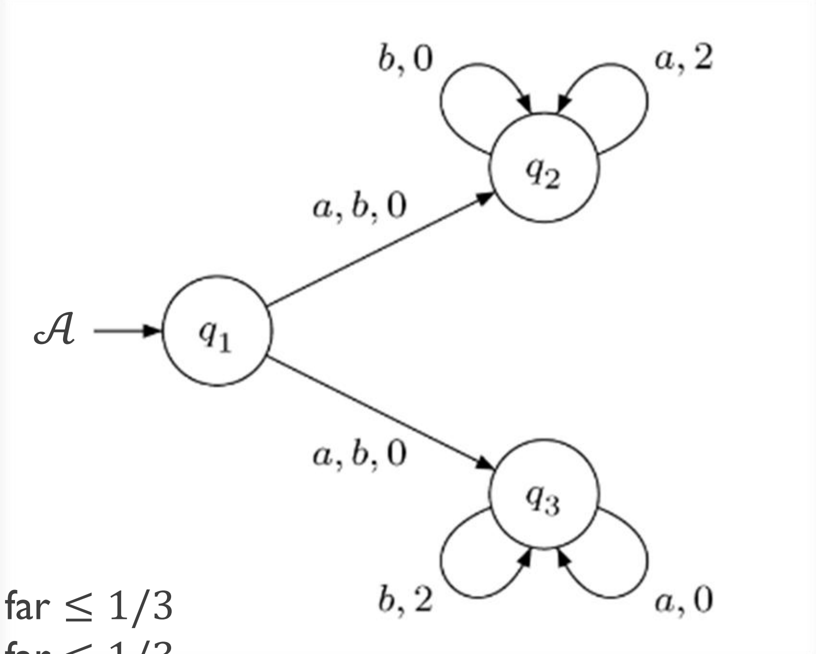
Approximating $\perp_{\mathcal{A}}$ by Lasso Words

Why it fails for LimInfAvg [CDH08]

- $\mathcal{A}(w) \geq 1$ for all lasso w

- $\mathcal{A}(w') < 1$ for some non-lasso w'

$$w' = u_0 u_1 u_2 \dots$$
$$u_0 = \epsilon$$
$$u_i = \alpha_i \beta_i$$



α_i: sufficiently long block of a's s.t. the b-density so far $\leq 1/3$

β_i: sufficiently long block of b's s.t. the a-density so far $\leq 1/3$

$$\Rightarrow \mathcal{A}(w') \leq 2/3$$

Chatterjee, Doyen, and Henzinger, "Quantitative languages", CSL 2008.

Approximating $\bot_{\mathcal{A}}$ by Lasso Words

Why it works for $\mathrm{LimSupAvg}$

- Given a word w with value $< \bot_{\mathcal{A}} + \frac{\varepsilon}{4}$,
  all sufficiently long prefixes have every run's average $< \bot_{\mathcal{A}} + \frac{\varepsilon}{2}$.
  (Otherwise, contradiction to word's value)
- Pick two prefixes $u_1 \prec u_2 \prec w$ reaching the same state set,
  and pump the middle.
- Any run on the pumped block stays low average,
  so the lasso's $\mathrm{LimSupAvg}$ stays $<\bot_{\mathcal{A}} + \varepsilon$.
- $\Rightarrow$ For QLAs with nondeterministic $\mathrm{LimSupAvg}$-automata,
  unrestricted $\geq$-universality = finite-state $\geq$-universality

Approximating $\perp_{\mathcal{A}}$ by Lasso Words

Corollary.

Let $f \in \{\text{LimSupAvg}, \text{LimInfAvg}\}$.

Given $\mathcal{A} = (\text{Sup}, f, \mathcal{T})$ and $k \in \mathbb{Q}$,
checking whether $\mathcal{A}(w) \geq k$ for every lasso word w is undecidable.

$\{\text{LimSupAvg}, \text{LimInfAvg}\} \times \{\text{All words}, \text{Lasso words}\} \times \{>, \geq\}$

- Every other combination here was proved undecidable [DDG+10, HPP+18]

Degorre, Doyen, Gentilini, Raskin, and Toruńczyk, "Energy and mean-payoff games with imperfect information", CSL 2010.
Hunter, Pauly, Pérez, and Raskin, "Mean-payoff games with partial observation", TCS, 2018.

Evaluation

Given a QLA $\mathbb{A} = (h, \mathcal{A})$ and an NBA $\mathcal{B}$.

Compute $\mathbb{A}(L_\mathcal{B})$.

Is there a word w such that $\mathbb{A}(L_\mathcal{B}) = \mathcal{A}(w)$?

$f \in \cdots$	$\{\mathrm{Inf}, \mathrm{Sup}, \mathrm{LimInf}, \mathrm{LimSup}\}$	$\{\mathrm{LimInfAvg}, \mathrm{LimSupAvg}\}$	$\{\mathrm{DSum}\}$
	$w \in L_\mathcal{B}$ and lasso	$w \in L_\mathcal{B}$ when $g = h$	$w \in \mathrm{Safe}(L_\mathcal{B})$ when $g = h$
	PTime when $g = h$ PSpace when $g \neq h$	PTime when $g = h$ Incomputable when $g \neq h$	PTime when $g = h$ Open-hard when $g \neq h$